Conditional stochastic processes applied to wave load predictions

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Abstract. The concept of conditional stochastic processes provides a powerful tool for evaluation and estimation of wave loads on ships and offshore structures. The paper will first consider conditional waves with focus on critical wave episodes. Then the inherent uncertainty in the results will be illustrated with an application where measured wave responses are used to predict the future variation in the responses within the next 5-30 seconds.

The main part of the paper is devoted to the application of the First Order Reliability Method (FORM) for derivation of critical wave episodes for different non-linear wave-induced responses. A coupling with Monte Carlo simulations is shown to be able to give uniform accuracy for all exceedance levels with moderate computational time even for rather complex non-linear problems. The procedure is illustrated by examples dealing with overturning of jack-up rigs, parametric rolling of ships, and slamming and whipping vibrations.

1. Introduction

Waves and wave loads at sea can only be described in terms of statistical quantities due to the inherent stochastic nature of ocean waves. The mathematical theory of stochastic processes provides means for the analysis. One area is the analysis of measured or calculated time records, where extreme value predictions can be made using various extrapolation procedures like the peak-over-threshold method and least-square fitting to Gumbel and Weibull distributions; see e.g. *Andersen and Jensen* (2014) for an analysis of full scale measurements.

However, when dealing with estimation of wave or response predictions given some prior information the theory of conditional processes is a powerful tool. Some examples are:

- The variation around a known value, e.g. a measured peak
- The short-term forecasting of a response, knowing previous values
- The probability of exceeding a prescribed response value
- Definition of critical wave scenarios for CFD calculations

The aim of the paper is to illustrate these possibilities with examples covering waves, ships and offshore structures. Throughout the paper it is assumed that the stochastic processes are stationary in a stochastic sense with unconditional zero mean values. The first chapter deals with linear and slightly non-linear responses whereas the last chapter considers fully non-linear wave response problems.

2. Gaussian and slightly non-Gaussian processes

The paper by *Tromans et al.* (1991) dealing with a definition of a design wave based on a known spectral density $S_x(\omega)$ of the waves initiated a significant research effort aiming at identification of response-specific critical wave sequences for application in model tests and computer simulations, see *Clauss* (2008) for a detailed overview. Within linear (wave) theory, the Slepian process, *Lindgren* (1970), provides the necessary theoretical framework. Thus, the conditional probability for the response value *X(t)* (e.g. wave elevation) as function of time *t* given its value x_0 at, say, *t*=0 together with its first, $\dot{x}_0$, and second derivative, $\ddot{x}_0$, with respect to time *t* at this point becomes Gaussian distributed with mean value $\mu(t)$ and variance $v(t)$:

$$\mu(t) = E\left[X(t)\middle|X(0)=x_0, \dot{X}(0)=\dot{x}_0, \ddot{X}(0)=\ddot{x}_0\right] = x_0\frac{\alpha r(t)+u(t)}{\alpha-1} + \dot{x}_0\sqrt{\frac{m_0}{m_2}}s(t) + \ddot{x}_0\frac{m_0}{m_2}\frac{r(t)+u(t)}{\alpha-1}$$
$$v(t) = m_0\left\{1-r^2(t)-s^2(t)-\frac{\left(r(t)+u(t)\right)^2}{\alpha-1}\right\} \qquad (1)$$

where a super-dot denotes differentiation with respect to time. The conditional mean and variance are expressed solely in terms of the normalized time-dependent autocorrelation function $r(t)$ and its first and second time derivative:

$$r(t) = \frac{E\left[X(0)X(t)\right]}{m_0}; \quad s(t) = \frac{E\left[\dot{X}(0)X(t)\right]}{\sqrt{m_0 m_2}} = -\sqrt{\frac{m_0}{m_2}}\dot{r}(t); \quad u(t) = \frac{E\left[\ddot{X}(0)X(t)\right]}{m_2} = \frac{m_0}{m_2}\ddot{r}(t) \qquad (2)$$

together with the spectral moments

$$m_n = \int_0^\infty \omega^n S_x(\omega)d\omega; \quad \alpha = \frac{m_0 m_4}{m_2^2} \qquad (3)$$

As $r(t)$ and $u(t)$ are nearly equal but with opposite sign, the inclusion of the second derivative is only important if α is close to, but not equal to 1, *Andersen et al.* (2013). For $\alpha \to 0$, Eq. (1) reduces to the deterministic result: $X(t) = x_0\cos(\varpi t) + \dot{x}_0\sin(\varpi t)/\varpi$.

Due to the Gaussian behavior the conditional mean value $\mu(t)$ is also the most probable value and can therefore been considered as representing the most probable shape of the response around a given peak, *Tromans et al.* (1991):

$$\mu(t) = E\left[X(t)\middle|X(0)=x_0, \dot{X}(0)=0\right] = x_0 r(t) \qquad (4)$$

neglecting conditioning on the second derivative. This result can be generalized to the conditional mean value $\mu_y(t)$ of a Gaussian response $Y(t)$ (with spectral density $S_y(\omega)$) given again that the correlated Gaussian process $X(t)$ has a known peak value x_0 at *t*=0, e.g. *Jensen* (2005):

$$\mu_y(t) = E\left[Y(t)\middle|X(0)=x_0, \dot{X}(0)=0\right] = \sigma_y u_0 r_{xy}(t)$$
$$r_{xy}(t) = E\left[Y(t)X(0)\right]/(\sigma_y\sigma_x) \tag{5}$$
$$\sigma_x^2 = E\left[X^2\right] = m_0; \quad \sigma_y^2 = E\left[Y^2\right] = \int_0^\infty S_y(\omega)d\omega; \quad u_0 = x_0/\sigma_x$$

This is a useful expression when e.g. the wave kinematics given a wave crest is needed.

For slightly non-Gaussian processes the conditional mean response can be written as, *Jensen* (2005),

$$\mu_y(t) = E\left[Y(t)\middle|X(0)=x_0, \dot{X}(0)=0\right] =$$
$$\sigma_y u_0 r_{xy}(t) + \sigma_y\left(\frac{1}{2}(u_0^2-1)\ \ (\lambda_{201}(t) - r_{xy}(t)\lambda_{300}) - \frac{1}{2}\left(\lambda_{021}(t) - r_{xy}(t)\lambda_{120} - s_{xy}(t)\lambda_{030}\right)\right) \tag{6}$$
$$s_{xy}(t) = E\left[Y(t)\dot{X}(0)\right]/\left(\sigma_y\sigma_{\dot{x}}\right) = -\dot{r}_{xy}(t)\frac{\sigma_x}{\sigma_{\dot{x}}}; \quad \sigma_{\dot{x}}^2 = E\left[\dot{X}^2\right] = m_2$$

where λ_{ijk} are normalized spectral moments defined as

$$\lambda_{klm}(t) = E\left[X^k(0)\dot{X}^l(0)Y^m(t)\right]/\left(\sigma_x^k\sigma_{\dot{x}}^l\sigma_y^m\right) \tag{7}$$

Higher order terms can be included, *Jensen* (2005), but are seldom useful for waves and wave loads due to lack of accurate estimations of the higher order spectral moments.

An example of the application of Eq. (6) to wave elevations is shown in Fig. 1. Here an attempt to model the so-called Draupner New Year Wave, *Haver and Andersen* (2000), is made using linear and second-order Stokes wave theory. The results are shown both for long-crested and short-crested waves with a cosine-square spreading function and both for deep water waves and for the actual water depth (70 m) at the location of the Draupner platform. All calculations are conditional on the measured wave crest of 18.5 m and based on a Pierson-Moskowitz wave spectrum with significant wave height H_s=12 m and mean zero up-crossing period T_z=12 sec. As the measurement is just one possible outcome of the wave scenario any strict agreement with the calculated conditional mean wave cannot of course be expected, but overall the second-order model gives a better estimate of the variation of the wave elevation in the vicinity of the given crest value. Further details can be found in *Jensen* (2005), including conditional mean horizontal wave particle velocities. However, no measurements of the wave kinematics were made.

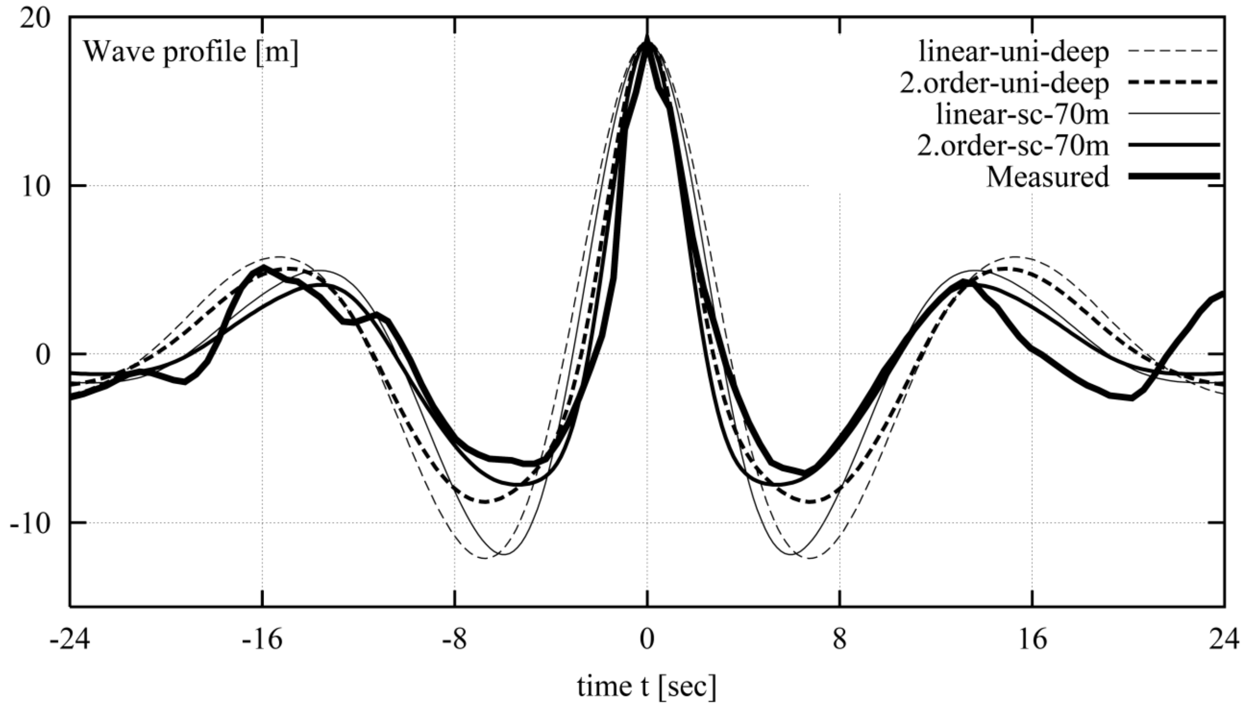

Figure 1. Draupner New Year Wave elevation. Measurements from *Haver and Andersen* (2000). Conditional mean wave elevations based on H_s = 12 m, T_z = 12 sec., no current. Uni: long-crested, sc: short-crested waves. Deep water and finite water depth *h* = 70 m. *Jensen* (2005).

2.1 Critical wave episodes

By prescribing a specific value of an input process $X(t)$, e.g. the wave elevation, in Eq. (6) associated conditional mean values of any correlated responses $Y(t)$ can be found. However, often another application where $X(t)$ is a wave response and $Y(t)$ the associated wave elevation is very useful as it determines the most probably wave episode $Y(t)$ yielding a prescribed peak response at *t*=0: $X(0) = x_0$, $\dot{X}(0) = 0$. This was first investigated by *Adegeest et al.* (1998) for wave bending moments in ships and later by many others, e.g. *Cassidy et al.* (2001), *Dietz et al.* (2004). The idea is that these critical wave episodes then can be applied as input to time-domain simulations using time-consuming non-linear hydrodynamic models as a mean to reduce the necessary length of the time series. The probability of exceedance follows directly from the Gaussian property of $X(0): P\left(X(0) > x_0\right) = \Phi\left(-x_0 / \sigma_x\right)$, where Φ is the standard normal distribution function.

The non-linear peak response $x_{0,non-linear}$ obtained from these simulations can then be considered as a transformation of the linear peak response x_0 and, if monotonic; the associated statistics is simply obtained by associating the non-linear peak value $x_{0,non-linear}$ with the same probability as the corresponding linear value x_0. Calculations with different values of x_0 then give the full distribution of the non-linear response. For instance, the mean up-crossing rate $\nu^+(x_{0,non-linear})$ becomes:

$$\nu^+(x_{0,non-linear}) = \nu^+(0)\exp\left(-\frac{x_0^2(x_{0,non-linear})}{2m_0}\right) \tag{8}$$

However, this definition of a critical wave episode only represents the mean response around a given maxima. Hence, memory effects might not be sufficiently accounted for. To circumvent this problem the critical wave episode can be superimposed with realizations of the unconstrained stochastic wave*, Lindgren* (1970). A set of realization of this combined wave is then used as input to the non-linear hydrodynamic calculations. The result is a distribution of the non-linear response, conditional on a given linear response peak x_0. Un-conditioning with respect to the linear maxima can then be performed, assuming that the linear peak response is Rayleigh distributed, *Dietz et al.* (2004), *Oberhagemann et al.* (2012). This method is called a Conditional Random Response Waves (CRRW) approach.

Another procedure has been suggested in *Alford et al.* (2011). Here the phase angles in the stochastic wave representation are estimated such that a given wave or response event is obtained. Monte Carlo simulations of moderately rare events of a random process indicate that the random Fourier component phases are non-uniform, non-identically distributed, and dependent on the rarity of the target event. A Modified Gaussian distribution has been found to compare reasonable well with the simulations and is used in the so-called Design Load Generator to generate design time series with a given expected value at a specific time.

The procedures outlined above rely on the assumption that the linear result yields a critical wave scenario which also holds for the real non-linear problem in question. Thus for instance the non-linear response should show a peak value at nearly the same point in time as the linear problem when using the same critical wave episode. Otherwise, the linear constraint model, Eq. (6), might not capture the effect of non-linearity correctly. One extreme example is parametric rolling of ships, where no linear solution exists due to the bifurcation type of response. In such cases, a procedure based on the First Order Reliability Method (FORM) provides a good alternative, especially if coupled with Monte Carlo Simulations (MCS). This will be discussed in Chapter 3, but before another application of Eq. (1) is considered below.

2.2 Short-term forecasting of measured responses

Provided a measured wave-induced response shows a Gaussian behaviour Eq. (1) can be used to estimate its future value within the time frame of the memory effect of the response, typically of the order of 10-20 sec. This has been investigated in *Andersen et al.* (2013) using hull girder strain measurement on-board a large container vessel. As the estimated response $X(t)\big|X(0)=x_0, \dot{X}(0)=\dot{x}_0, \ddot{X}(0)=\ddot{x}_0$ is Gaussian distributed the variance $v(t)$ determines completely its statistical variations with respect to time. From measurements the autocorrelation function *r(t)*, Eq. (2), can be estimated. The results show that the variance is increasing quite rapidly with time at the beginning and levels out to the un-constraint variance m_0 around 20 sec forward of the measurement, Fig. 2. Waviness in the variation is also noticed due to the variation in the autocorrelation *r(t)* of the signal with the dominant period in the response. The rapid increase in variance with time makes the short term predictions rather uncertain already after about 5 sec. Furthermore, a knowledge of the second derivative of the response at *t*=0 has little influence on the result.

The results can be improved if conditioning is made of more measured values prior to *t*=0. Hence, Eq. (1) is replaced by, ignoring conditioning of any derivatives:

$$\begin{aligned} \mu_n(t) &= E\left[X(t)\big|X(0)=x_0, X(t_1)=x_1,\ldots, X(t_n)=x_n\right] = \underline{r}^T(t)\underline{\underline{R}}^{-1}\underline{x} \\ v_n(t) &= m_0\left(1-\underline{r}^T(t)\underline{\underline{R}}^{-1}\underline{r}(t)\right) \end{aligned} \tag{9}$$

with $0 > t_1 > t_2 > .. > t_n$ and

$$\begin{aligned} \underline{r}(t) &= \left[r(t), r(t-t_1), r(t-t_2),\ldots, r(t-t_n)\right]^T \\ \underline{x} &= \left[x_0, x_1, x_2,\ldots, x_n\right]^T \\ \underline{\underline{R}} &= \begin{bmatrix} 1 & r(t_1) & r(t_2) & \ldots\ldots\ldots & r(t_n) \\ & 1 & r(t_2-t_1) & \ldots\ldots. & r(t_n-t_1) \\ & & 1 & \ldots\ldots. & r(t_n-t_2) \\ & & & \ldots\ldots. & \\ & & & & 1 \end{bmatrix} \end{aligned} \tag{10}$$

Thereby the accuracy of the short term predictions increases as the variance is reduced as shown in Fig. 2 for *n*=10 and $t_i - t_{i+1} = 1$ sec.

Finally, it is noted that the result, Eq. (10), based on conditional stochastic processes is exactly the same as the Autoregressive Predictor (AR) method in the Yule-Walker formulation:

$$\left.\begin{aligned} &\text{AR: } x(t) = \underline{a}(t)^T\,\underline{x};\, t>0 \\ &\text{Yule-Walker: } \underline{a}(t) = \underline{\underline{R}}^{-1}\underline{r}(t) \end{aligned}\right\} \Rightarrow x(t) = \underline{r}(t)^T\,\underline{\underline{R}}^{-1}\underline{x} = \mu_n(t) \tag{11}$$

Generally, the accuracy decreases with the bandwidth of the response. Usually the time increment $t_i - t_{i+1}$ should be around 0.5 to 1 sec and the number of terms, *n*, enough to cover the memory period, i.e. *n* =30 to 40, e.g. *Fusco and Ringwood* (2010).

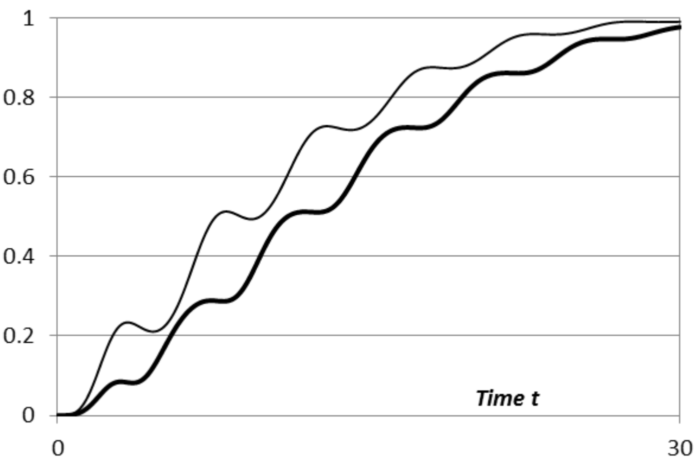

Figure 2. Hull girder strain measurements. Normalized variances $v(t)/m_0$. Thin line: Eq. (1), Thick line: Eq.(10) with *n*=10, *Andersen et al. (2013)*

3. Non-linear processes

The application of Eq. (5) or Eq. (6) for definition of a response-specific critical wave episode $Y(t)$ relies as mentioned before on its ability to describe also the most critical wave episode when a more accurate non-linear model of the wave-induced response is applied. Thus, for instance Eq.(6) is able to capture the slight non-linearity seen in the wave-induced bending moment in container ships sailing in moderate seaways, e.g. *Jensen and Pedersen* (1979). However, if slamming plays an important role then full CFD calculations might be needed, e.g. *Oberhagemann et al.* (2008) and *Seng et al.* (2012). Other examples with strong non-linearity are overturning of jack-up rigs and parametric rolling of ships, to name just a few. Common for all these problems is that a critical wave scenario calculated from Eqs. (5)-(6) does not give a good representation of the most probable wave episode for the non-linear problem.

One solution is to apply direct Monte Carlo Simulations (MCS), but usually the computational effort needed for extreme value predictions is prohibitive large. An alternative is to look at approximate procedures, which focuses on the tail in the probability distribution. A good candidate is the First Order Reliability Method (FORM), widely used in structural reliability analyses, but also an efficient method for extreme value predictions, as suggested originally by *der Kiureghian* (2000) and derived for wave loads by *Jensen and Capul* (2006). Due to efficient optimisation procedures implemented in standard FORM codes and the short duration of the time domain simulations needed (typically 60-300 sec. to cover the hydrodynamic memory effects in the response) the calculation of the mean up-crossing rate of the response is rather fast, implying that fairly complicated non-linear effects can be included.

In the FORM procedure a limit state surface G is defined as:

$$G(\underline{u}) \equiv x_0 - X(0|\underline{u}) = 0 \tag{12}$$

where $\underline{u}^T = [u_1, u_2, ,..., u_n]$ are uncorrelated standard normal distributed variables defining the stochastic variations of the input $Y(t|\underline{u})$, e.g. the wave elevation. Realizations of the associated wave response $X(t|\underline{u})$ are determined by a time domain analysis covering $-t_0 \le t \le 0$. For a stationary stochastic process the value of t_0 is unimportant as long as it is longer than the memory in the system, i.e. the influence at $t=0$ of the initial conditions at $t=-t_0$ has vanished. The realisation, which reaches the given threshold (peak) x_0 at time $t=0$ with the highest probability, is sought. The corresponding values $\underline{u}^*$ of $\underline{u}$ is denoted the design point and uniquely defines the deterministic input process $Y^*(t) = Y(t|\underline{u}^*)$ which, due to the normal distributions of $\underline{u}$, also is the most probable input process leading to the desired response x_0 at $t=0$. Hence, it can be considered as the best choice for a deterministic critical input process, e.g. a critical wave episode.

The value of $\underline{u}^*$ is the solution to the optimization problem:

$$\underline{u}^* : \text{ Minimize } \underline{u}^T\underline{u}; \text{ Subject to } G(\underline{u}) = 0 \tag{13}$$

The probability $P(X(0) > x_0) = P(G < 0)$ cannot in general be calculated as it depends on the non-linearity in the response $X(t|\underline{u})$. Instead, it is approximated by the probability of exceedance the limit state surface function, linearized around the design point:

$$G(\underline{u}) \approx G(\underline{u}^*) + \left(\left. \frac{\partial G}{\partial \underline{u}} \right|_{\underline{u}=\underline{u}^*} \right)^T \left(\underline{u} - \underline{u}^* \right) = 0 \tag{14}$$

The design point vector $\underline{u}^*$ is normal to the linearized limit state surface and measures the shortest distance from origin to the limit state surface. It can therefore from pure geometrical considerations be written as

$$\underline{u}^* = x_0 \frac{\underline{e}}{\|\underline{e}\|^2} \tag{15}$$

The normal vector $\underline{e}$ is given as

$$\underline{e} = \frac{\partial X}{\partial \underline{u}}\bigg|_{\underline{u}=\underline{u}^*} = \left[e_1, e_2, ,..., e_n\right]^T ; \quad \underline{e}^T \underline{e} \equiv \left\|\underline{e}\right\|^2 \tag{16}$$

Note, that Eq. (15) is not an equation for determination of $\underline{u}^*$ as $\underline{e}$ depends on x_0 and $\underline{u}^*$, but merely just a way to write the result of Eq. (13). As $G(\underline{u}^*) = 0$ according to Eq. (13) the linearized limit state surface, Eq. (14) can be written

$$x_0 - \underline{e}^T \underline{u} = 0 \tag{17}$$

Because the linearization is around a design point usually associated with a low probability of occurrence, this linearization has been denoted Tail-equivalent linearization, *Fujimura and der Kiureghian* (2007).

Due to the linearization, the statistics of the response $X(0)$ follows from Gaussian processes. The variance

$$m_0 = E\left[X^2(0)\right] \approx E\left[\left(\underline{e}^T \underline{u}\right)^2\right] = \underline{e}^T \underline{e} \equiv \left\|\underline{e}\right\|^2 \tag{18}$$

and, hence

$$P\left(X(0) > x_0\right) \approx P\left(\underline{e}^T \underline{u} > x_0\right) = \Phi\left(-\beta_{FORM}\right) \tag{19}$$

Here β_{FORM} is the reliability index, defined as shortest distance to the limit state surface:

$$\beta_{FORM} = \sqrt{\underline{u}^{*T} \underline{u}^*} = \frac{x_0}{\left\|e\right\|} \tag{20}$$

For extreme value estimations other statistical measures as for instance the mean up-crossing rate

$$v^+(x_0) = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}} \exp\left(-\frac{{x_0}^2}{2m_0}\right) \tag{21}$$

are more useful. It requires calculation of the spectral moment m_2 and therefore also a spectral formulation of the response. As shown in *Jensen* (2011) the spectral density $S_x(\omega)$ of the linearized response becomes

$$S_x(\omega_i) d\omega = \left(\frac{\partial X}{\partial u_i}\bigg|_{\underline{u}=\underline{u}^*}\right)^2 = e_i^2\left(\omega_i\right) \tag{22}$$

where each u_i is associated with a sinusoidal linearized response variation in time with frequency ω_i. Hence,

$$m_2 = \int \omega^2 S_x(\omega) d\omega = \sum_{i=1}^{n} \omega_i^2 e_i^2 \tag{23}$$

It is noted that both m_0 and m_2 depend on the design point and thereby x_0 through $\underline{e}$.

In summary, Eq. (6), is now replaced by

$$\mu_y\left(t\right) = Y^*(t) = Y(t|\underline{u}^*) \tag{24}$$

with the statistics given by Eqs. (18)-(23). For a linear system: $Y(t) = \underline{a}(t)^T \underline{u}, \quad X(0) = \underline{e}^T \underline{u}$,

$$Y^*(t) = \underline{a}(t)^T \underline{u}^* = x_0 \frac{\underline{a}(t)^T \underline{e}}{\left\|\underline{e}\right\|^2} = x_0 \frac{E\left[Y(t)X(0)\right]}{m_0} \tag{25}$$

i.e. Eq. (5) as expected.

An important observation can be found from Eq. (17). Consider the case where the stochastic parameters $\underline{u}$ all are scaled by the same deterministic parameter c and that c does not appear anywhere else in the hydrodynamic model. The obvious example is the significant wave height in a spectral model of the wave elevation using e.g. a Pierson-Moskowitz or JONSWAP spectrum. Then the solution to Eq. (13) will simply be $\underline{u}_c^* = \underline{u}^* / c$ where $\underline{u}^*$ is the solution with $c = 1$. As $c\underline{u}_c^* = \underline{u}^*$, the most probable input process $Y(t|\underline{u}^*)$ leading to a given response peak x_0 remains unchanged, only the probability $P\left(X(0) > x_0\right) \approx \Phi\left(-\beta_{FORM}\right)$ that it occurs changes as

$$\beta_{FORM}(x_0, c) = \sqrt{\underline{u}_c^{*T} \underline{u}_c^*} = \frac{\sqrt{\underline{u}^{*T} \underline{u}^*}}{c} = \beta_{FORM}(x_0, 1) \tag{26}$$

This property is very useful as for instance to perform simulations in artificial high sea states (high value of *c*) where the occurrence of $X(0) > x_0$ happens quite often and then afterwards scale the probability down to more realistic sea states. It should be noted that this behavior has been derived from purely physical and ingenious arguments by *Tonguc and Söding* (1986), long before the FORM formulation was applied to this problem. The scaling property is not limited to wave loads, but can be applied to any load scenario where the intensity of all load components scales with the same factor. An example with both wind and wave loads acting on a floating offshore wind turbine is given in *Jensen et al.* (2011). In the following sections various examples of the application of the FORM procedure is discussed and illustrated.

3.1 Overturning of a jack-up rig

In the site assessment analysis of jack-up rigs, global responses such as air gab, overturning moment, base shear and horizontal deck sway play an important role. The calculations are usually done with the wave forces modelled by Morison's equation and with integration to the instantaneous water surface. The lowest natural period of a large rig is typically around 5-8 sec and therefore dynamic effects must be taken into account in the structural response. The ocean waves are modelled by stochastic waves specified by a wave spectrum in each stationary sea state to obtain realistic modelling of the dynamic amplification. In the following, the results of a FORM analysis and Monte Carlo simulations to the overturning of a jack-up rig are discussed. Thus, the limit state surface G is taken as the difference between the restoring moment and wave-induced overturning moment. The results are taken from *Jensen* (2011) where further details can be found.

First a FORM analysis has been performed using both a linear and a second order model of the stochastic waves.

Stationary sea conditions are assumed and specified by a standard JONSWAP wave spectrum in terms of a significant wave height H_S and zero-crossing period T_Z. Long-crested waves are considered travelling towards the up-wave leg. Pinned bottom support and rigid leg/hull connections are assumed. The wave spectral density is divided into 15 equidistant spaced components yielding n=30 uncorrelated standard normal distributed variables $\underline{u}$, each defining a sinusoidal variation of the waves. The time domain simulations in the FORM analysis, Eqs.(12)-(13), are carried out over a length of t_0 = 60 sec as the effect of the initial conditions are found negligible after 60 sec. Fig. 3 shows the results as function of $1/H_S$ for a zero-crossing wave period T_Z =9 sec, where dynamic effects becomes important as the lowest natural period of the rig is 8.3 sec.

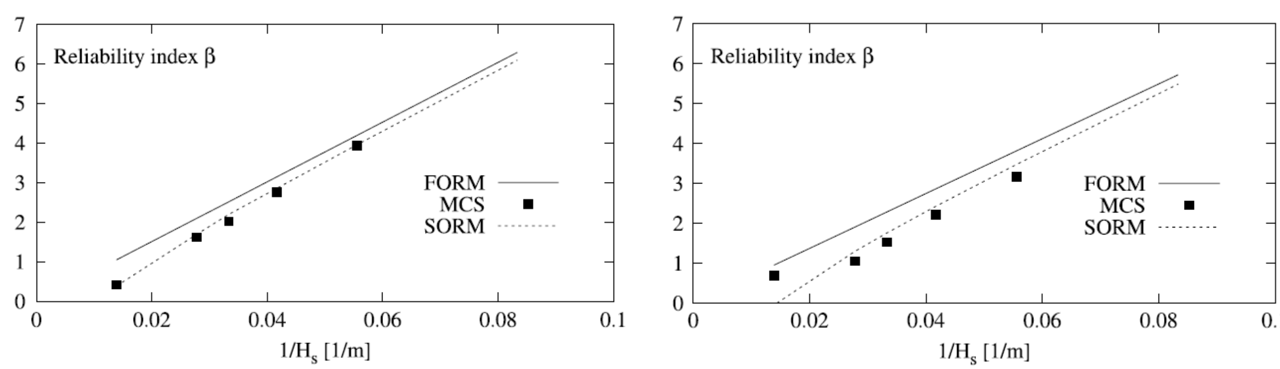

Figure 3. FORM, SORM and Monte Carlo simulation results as function of 1/H_s. Linear (left) and Second order (right) waves and T_z=9 sec. *Jensen (2011)*.

It is seen that inclusion of second order wave components reduce the reliability index mostly due to the higher crest values in second order waves coupled with the wave load integration to the instantaneous water surface.

The calculations have also been done for an improved reliability procedure, The Second Order Reliability Method (SORM), where the curvature of the limit state surface around the design point is taken into account in the estimation of $P\left(X(0) > x_0\right) \approx \Phi\left(-\beta_{SORM}\right)$.

Only one FORM calculation is performed for each of the two cases as the reliability index β_{FORM} is exactly inversely proportional to H_S as discussed above. This property does not hold for the SORM analyses, so here calculations are done for different values of H_S. Fig. 3 also includes results from direct Monte Carlo simulations (MCS) done in order to validate the FORM and SORM results. The SORM results are in better agreement with the MCS results, but in general this not always so. The FORM procedure is asymptotically correct for very low probabilities of exceedance, i.e. high values of the reliability index, in agreement with the results in Fig. 3. Due to the high computational effort required, MCS (with COV=0.05) has only been done for sea states with unrealistic high significant wave height ($H_S \geq$ 18 m). The important observation here is the same as found in *Tonguc and Söding* (1986) by other arguments, namely that a scaling of the reliability index and thereby the probability of exceedance as suggested by the FORM results provides a rather accurate way to scale the calculations down to realistic sea states with H_S around 10-15 m. Thus, simulations can conveniently be done in unphysical high sea states with a corresponding high probability of occurrence of the requested response and thereby of shorter duration. A modification of the scaling to read

$$\beta^2 = A + BH_S^{-2} \tag{27}$$

has been suggested by *Shigurov et al.* (2010), whereas $\beta = \beta_{FORM} + CH_S$ is proposed in *Jensen* (2011), both with the aim to add flexibility in the curve fitting of the MCS results. Especially, the combination of Monte Carlo simulations using artificially increased significant wave heights combined with a FORM calculation seems to be able to provide a rather uniform accuracy over a large range of exceedance levels, e.g. Fig. 3. *Papanikolaou et al.* (2014) suggest the same procedure in a study of bow acceleration.

3.2 Parametric rolling of ships

Parametric rolling of ships has rather recently attached a significant research effort due to several possible full scale experience of this much undesired behavior of ships at sea. Loss of containers typically becomes the results with severe economic consequences. The literature on parametric rolling is large; see e.g. *France et al.* (2003), *Shin et al.* (2004) and *Bulian* (2005). The crucial term in the governing equations of motion responsible for parametric rolling is the time-varying restoring moment, but other terms coupling the heave and the pitch motions to roll are also important. A versatile, yet tractable, model of parametric rolling has been developed by *Kroeger* (1986). The heave, pitch, and yaw motions are determined by standard strip theory formulations, whereas the surge motion is calculated from the incident wave pressure distribution. The advantage of this formulation compared to full non-linear calculations is the much faster computational speed, while still retaining a coupling between all six degrees of freedom, *Krüger et al.* (2004). This model with some simplifications has been applied in FORM analysis, Jensen (2007) in order to investigate whether a critical wave episode can be extracted for this phenomenon. The limit state surface is taken as a prescribed roll angle and the instantaneous GZ curve is determined using an equivalent wave procedure somewhat similar to *Kroeger* (1986).

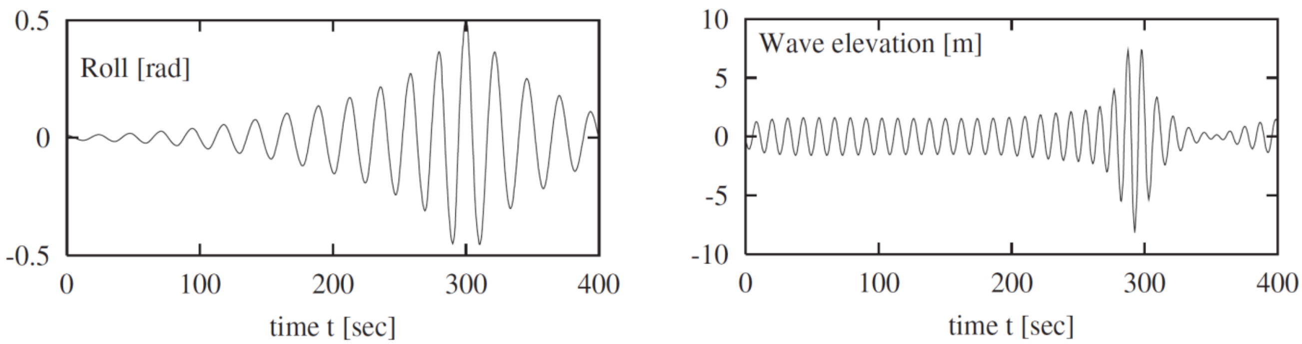

Figure 4. Parametric rolling of a ship. Left: Most probable roll response leading to a roll angle of 0.5 rad. Right: The corresponding most probable wave episode amidships, *Jensen* (2007). Note: Initial conditions at *t*=0 and prescribed response at t=300 sec.

Fig. 4 shows results for the critical wave episode $Y(t|\underline{u}^*)$ leading to $X(300\,\mathrm{sec}|\underline{u}^*) = 0.5\,\mathrm{rad}$. The memory in parametric rolling is much larger than in most other ship motion components. At least $t_0 =$ 300 sec is found necessary in the time domain simulations in the FORM analysis. The interesting observation is that the critical wave episode basically is a sum of two contributions: a 'regular' wave with encounter frequency close to twice the roll frequency and a wave height just enough to trigger parametric rolling and, a 'transient' wave with magnitude depending on the prescribed roll response. Thereby, this critical wave episode differs significantly from the usual form seen in linear and slightly non-linear problems, where only the transient part is present. The reason is that parametric rolling is a bifurcation type of problem needing a minimum level of excitation before onset can take place.

The scaling properties with significant wave height mentioned previously is still valid and comparisons with Monte Carlo simulations, *Jensen* (2007), show good agreement between the FORM and MCS results.

The non-linear hydrodynamic model used here in the FORM analysis for parametric rolling can of course be improved, but probably at the expense of much longer computational time. An alternative is to apply the critical wave episode $Y(t|\underline{u}^*)$ determined from the FORM analysis as input to a more accurate non-linear analysis. By assuming a monotonic relation between the specified response peak in the FORM analysis and the response peak obtained in the new simulation using the deterministic critical wave episode, the probability distribution of the improved result are obtained as described in Section 2.1:

$$\beta(x_{0,\text{improved}}) = \beta_{FORM}\left(x_0(x_{0,\text{improved}})\right) \tag{28}$$

This so-called model-correction-factor approach, *Ditlevsen and Arnbjerg-Nielsen* (1994), *Seng and Jensen* (2013), will be illustrated in the next section dealing with slamming loads on ships.

3.3 Slamming load prediction

Estimation of slamming loads in the bow of a vessel during operation in heavy sea constitutes a very difficult hydrodynamic problem. A detailed investigation of the pressures during bow submergence requires very accurate 3D hydrodynamic (CFD) models like a finite volume formulation with the free surface captured by a volume-of-fluid technique. Such formulation applied in a stochastic sea will generally be prohibitive expensive even on a high performance computer cluster (HPC). Therefore, the concept of a deterministic critical wave episode derived from a FORM analysis using a simpler non-linear model could be attractive as the use of this wave episode in a CFD calculation could increase the accuracy in the probability distribution of the slamming load using Eq. (28) with reasonable effort.

As an example, *Seng et al.* (2012), *Seng and Jensen* (2012) derived the critical wave episode for Panamax container ship using a time-domain non-linear strip theory calculation procedure, *Xia et al.* (1998), linked to a FORM code. The hull is assumed rigid. The vessel is sailing with 5 m/sec in head sea in a long-crested sea state characterized by a Pierson-Moskowitz wave spectrum with a significant wave height H_S = 10 m and a mean wave period of 11.35 sec.

The limit state surface is defined as the wave-induced sagging bending moment being equal to 3000 MNm. The results are shown in Fig. 5 as function of time until the maximum bending moment response occurred at the pre-selected time (50 s) after initiation of the calculations. This time distance is sufficiently long to ensure that transient (memory) effects from the arbitrary initial conditions in the simulations do not influence the maximum response.

The FORM reliability index for this event is 2.75 corresponding to a return period of approximately 10 minutes in this stationary sea state and operational condition. Thus, this is a rather frequent event with moderate non-linearity and nearly no slamming. It is for this situation a CFD analysis is performed and the results included in Fig. 5 (dashed lines in the left figure). Clearly a good agreement between the non-linear strip theory and a full 3D CFD calculation is found including the peak values. Thus a model-correction-factor approach could be used, but is barely necessary.

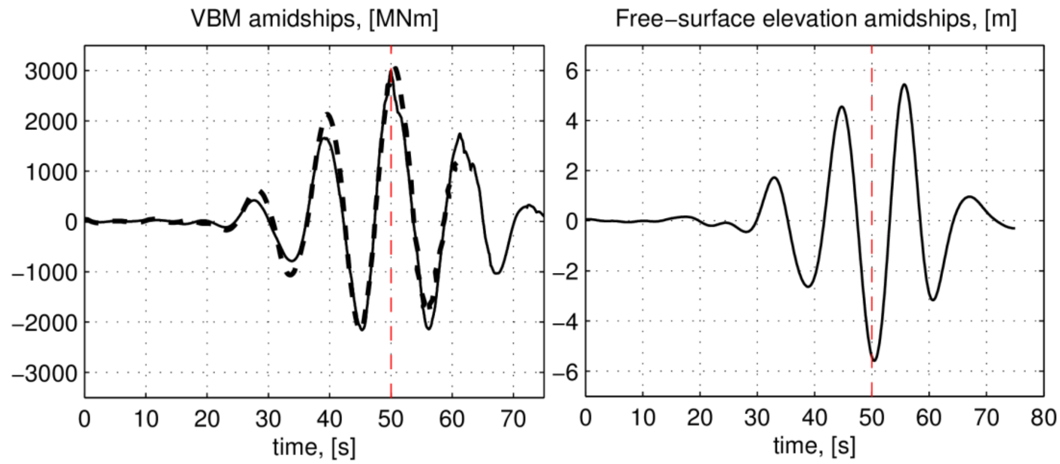

Figure 5. Wave bending moment in a Panamax container ship. Left: Most probable bending moment response leading to a sagging wave bending moment of 3000 MNm. Full lines: non-linear strip theory, dashed lines: CFD. Right: Corresponding most probable wave episode at amidships derived from the non-linear strip theory, *Seng and Jensen* (2012). Note: Initial conditions at t=0 and prescribed response at t=50 sec and, sag is positive.

The results for two other, more severe events are shown in Fig. 6 for another vessel. Here the differences are larger, but still the responses from the non-linear strip theory and the CFD calculation resemble each other so well that the probability estimate from the FORM analysis using the non-linear strip theory can be applied also to the CFD peak response with some confidence using Eq. (28). The peak values from the CFD calculations are seen to be considerable less than those from the non-linear strip theory calculation, probably due to the very coarse momentum slamming model in the strip theory approach.

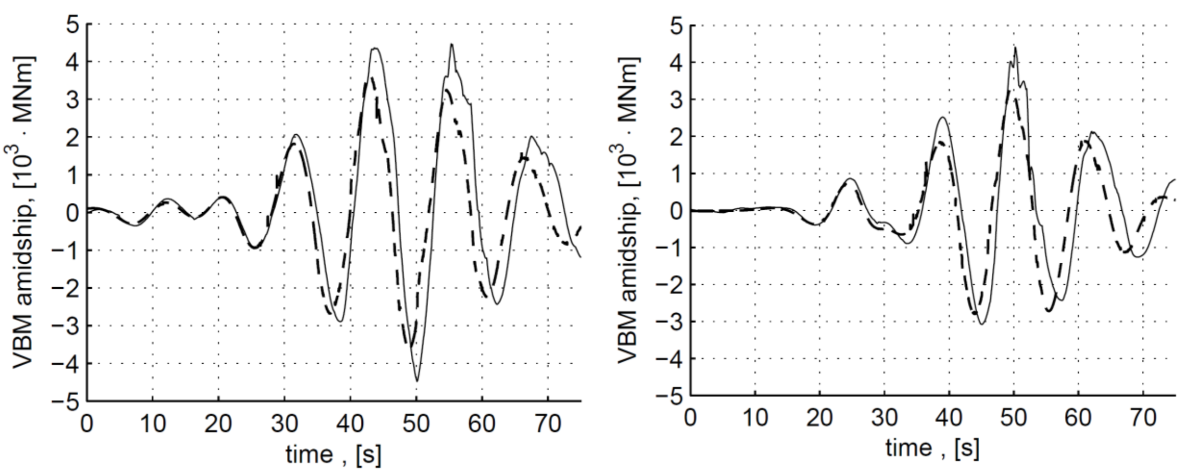

Figure 6. Wave bending moment in a post-Panamax container ship. Left: Most probable bending moment response leading to a hogging wave bending moment of 4500 MNm. Right: Most probable bending moment response leading to a sagging wave bending moment of 4500 MNm. Full lines: non-linear strip theory, dashed lines: CFD. *Seng and Jensen* (2012). Note: Initial conditions at t=0 and prescribed response at t=50 sec and, sag is positive.

3.4 Whipping vibrations

The last example deals with also with slamming, but now for a flexible hull. The ship is a post Panamax container vessel and the non-linear strip theory, *Xia et al.* (1998), is applied again, but now taking the hull flexibility into account, *Andersen and Jensen* (2012) and *Seng and Jensen* (2013), modelling the stiffness of the vessel as a non-prismatic Timoshenko beam. Thereby, the critical wave episode derived from the FORM approach changes rather much compared to the same wave for the ship modelled as rigid body.

The main problem encountered was that whereas the non-linear strip theory calculation found whipping to be important for the critical wave episode, the use of this wave episode as input to the CFD calculation did not give rise to any notable whipping event. Thus the non-linear strip theory seems not to be a good predictor for what happens with respect to whipping using a more accurate 3D hydrodynamic model.

One possible alternative solution procedure is Monte Carlo CFD simulations with artificially increased significant wave heights resulting in large exceedance levels, *Oberhagemann et al.* (2012). However, extensive computer resources are then needed. Another suggestion, *Oberhagemann et al.* (2012), is the Conditional Random Response Waves approach, *Dietz et al.* (2004), but as it relies on the linear conditional formulation, Eq. (5), the ability of this approach to capture strong non-linearity like slamming and subsequent whipping vibrations might pose a problem. A combination of these two approaches is suggested to be the most feasible procedure and some examples show CPU times of the order 50 days on a computer cluster with 1000 CPU cores.

Sclavounos (2012) have recently suggested an interesting idea able to reduce the number *n* of wave components in the input process. The idea is to replace the usual sum of sinusoidal components with a Karhunen-Loeve representation of the stochastic wave field. Due to the close resemblance of the individual components in this representation with the autocorrelation function, much lesser terms seem to be needed in the stochastic representation of the waves. This will greatly reduce the calculation time in the FORM and the Conditional Random Response Waves approach. This gain can be used to apply a more accurate hydrodynamic model making the resulting critical wave episode a better predictor for the results using a full non-linear model.

Further work is clearly needed to resolve this issue. This is especially so because recent full scale measurements, e.g. *Andersen and Jensen* (2014), have shown that slamming-induced hull girder vibratory bending stresses can be as large as the wave-induced (rigid body) stresses. Such large whipping contributions have not been foreseen in the current classification rules and might have been one of the reasons for the collapse of the container MOL Comfort, *ClassNK* (2014).

4. Conclusion

The aim of the paper is to give an overview of possible use of conditional stochastic processes to analyse efficiently linear and non-linear wave response problems for ships and offshore structures.

Especially the combined use of a FORM analysis or the Constraint Random Response Wave approach for very low exceedance levels coupled with Monte Carlo simulations using artificially increased significant wave heights seems able to provide a rather uniform accuracy over a large range of exceedance levels. Also the ability of FORM to solve bifurcation type problems like parametric rolling is worth mentioning. The computational time needed for determination of the design point in FORM depends very strongly on the number of wave components applied. Therefore, the possibility to use another stochastic wave formulation, the Karhunen-Loeve representation, where the usual sinusoidal wave function is replaced by a function better modelling the autocorrelation of the stochastic wave looks promising.

The computational time required is still extremely high if a complicated non-linear hydrodynamic model like CFD is needed to get accurate results. This is especially so for hydro-elastic problems like slamming and whipping where small inaccuracies might change the combined wave and whipping extreme loads significantly.

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